ZETA FUNCTIONS AND ASYMPTOTIC ADDITIVE BASES WITH SOME UNUSUAL SETS OF PRIMES

WILLIAM D. BANKS

ABSTRACT. Fix $\delta \in (0,1]$, $\sigma_0 \in [0,1)$ and a real-valued function $\varepsilon(x)$ for which $\overline{\lim}_{x\to\infty} \varepsilon(x) \leqslant 0$. For every set of primes $\mathcal{P}$ whose counting function $\pi_{\mathcal{P}}(x)$ satisfies an estimate of the form

$$\pi_{\mathcal{P}}(x) = \delta\,\pi(x) + O\big(x^{\sigma_0+\varepsilon(x)}\big),$$

we define a zeta function $\zeta_{\mathcal{P}}(s)$ that is closely related to the Riemann zeta function $\zeta(s)$. For $\sigma_0 \leqslant \frac{1}{2}$, we show that the Riemann hypothesis is equivalent to the non-vanishing of $\zeta_{\mathcal{P}}(s)$ in the region $\{\sigma > \frac{1}{2}\}$.

For every set of primes $\mathcal{P}$ that contains the prime 2 and whose counting function satisfies an estimate of the form

$$\pi_{\mathcal{P}}(x) = \delta\,\pi(x) + O\big((\log\log x)^{\varepsilon(x)}\big),$$

we show that $\mathcal{P}$ is an *exact* asymptotic additive basis for $\mathbb{N}$, i.e., for some integer $h = h(\mathcal{P}) > 0$ the sumset $h\mathcal{P}$ contains all but finitely many natural numbers. For example, an exact asymptotic additive basis for $\mathbb{N}$ is provided by the set

$$\{2, 547, 1229, 1993, 2749, 3581, 4421, 5281\ldots\},$$

which consists of 2 and every hundredth prime thereafter.

1. INTRODUCTION AND STATEMENT OF RESULTS

Let $\mathbb{N}$ denote the set of positive integers and $\mathbb{P}$ the set of prime numbers. Denote by $\pi(x)$ the prime counting function

$$\pi(x) := \#\{p \leqslant x : p \in \mathbb{P}\},$$

and for any given set of primes $\mathcal{P}$, put

$$\pi_{\mathcal{P}}(x) := \#\{p \leqslant x : p \in \mathcal{P}\}.$$

Given $\delta \in (0,1]$, $\sigma_0 \in [0,1)$ and a real function $\varepsilon(x)$ such that $\overline{\lim}_{x\to\infty} \varepsilon(x) \leqslant 0$, let $\mathscr{A}(\delta,\sigma_0,\varepsilon)$ denote the class consisting of sets $\mathcal{P} \subseteq \mathbb{P}$ for which one has an estimate of the form

$$\pi_{\mathcal{P}}(x) = \delta\,\pi(x) + O\big(x^{\sigma_0+\varepsilon(x)}\big), \tag{1.1}$$

where the implied constant may depend on $\mathcal{P}$. Let $\mathscr{B}(\delta,\varepsilon)$ denote the class consisting of sets $\mathcal{P} \subseteq \mathbb{P}$ such that

$$\pi_{\mathcal{P}}(x) = \delta\,\pi(x) + O\big((\log\log x)^{\varepsilon(x)}\big), \tag{1.2}$$

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where again the implied constant may depend on $\mathcal{P}$. The aim of this paper is to state some general results that hold true for all sets in $\mathscr{A}(\delta,\sigma_0,\varepsilon)$, or for all sets in $\mathscr{B}(\delta,\varepsilon)$. We also give examples of sets $\mathcal{P}$ in these classes, to which our general results can be applied.

1.1. **Analogues of the Riemann zeta function.** The Riemann zeta function is defined in the half-plane $\{s=\sigma+it\in\mathbb{C}:\sigma>1\}$ by two equivalent expressions, namely

$$\zeta(s):=\sum_{n\in\mathbb{N}}n^{-s}=\prod_{p\in\mathbb{P}}(1-p^{-s})^{-1}\qquad(\sigma>1).$$

In the extraordinary memoir of Riemann **[19]** it is shown that $\zeta(s)$ extends to a meromorphic function on the complex plane, its only singularity being a simple pole at $s=1$, and that it satisfies a functional equation relating its values at s and $1-s$. The Riemann hypothesis (RH) asserts that every non-real zero of $\zeta(s)$ lies on the critical line $\{\sigma=\frac12\}$.

Although the function $\zeta(s)$ incorporates all of the primes into its definition, in this paper we observe that certain thin subsets of the primes also give rise to functions that are strikingly similar to $\zeta(s)$.

Theorem 1.1. *For any set $\mathcal{P}\in\mathscr{A}(\delta,\sigma_0,\varepsilon)$, the function $\zeta_{\mathcal{P}}(s)$ defined by*

$$\zeta_{\mathcal{P}}(s):=\prod_{p\in\mathcal{P}}(1-p^{-s})^{-1/\delta}\qquad(\sigma>1)$$

extends to a meromorphic function on the region $\{\sigma>\sigma_0\}$, and there is a function $f_{\mathcal{P}}(s)$ which is analytic on $\{\sigma>\sigma_0\}$ and has the property that

$$\zeta_{\mathcal{P}}(s)=\zeta(s)\exp(f_{\mathcal{P}}(s))\qquad(\sigma>\sigma_0).\tag{1.3}$$

This is proved in §2 below.

The following corollary is clear in view of (1.3); it shows that the truth of the Riemann hypothesis depends only on the distributional properties of certain (potentially thin) sets of primes.

Corollary 1.2. *If $\mathcal{P}\in\mathscr{A}(\delta,\sigma_0,\varepsilon)$ and $\sigma_0\leqslant\frac12$, then the Riemann hypothesis is true if and only if $\zeta_{\mathcal{P}}(s)\neq0$ in the half-plane $\{\sigma>\frac12\}$.*

Similarly, for every nontrivial primitive Dirichlet character χ, the Dirichlet L-function $L(s,\chi)$, which is initially defined by

$$L(s,\chi):=\sum_{n\in\mathbb{N}}\chi(n)n^{-s}=\prod_{p\in\mathbb{P}}(1-\chi(p)p^{-s})^{-1}\qquad(\sigma>1),$$

extends to an entire function on the complex plane and satisfies a functional equation relating its values at s and $1-s$. The generalized Riemann hypothesis (GRH) asserts that every non-real zero of $L(s,\chi)$ lies on the critical line.

The following result provides (in some cases) an analogue of Theorem 1.1. It is proved only for quadratic Dirichlet characters χ. For any such character, let us denote

$$\pi^-(x;\chi) := \#\{p \leqslant x : p \in \mathbb{P} \text{ and } \chi(p) = -1\},$$

and for a given set of primes $\mathcal{P}$, put

$$\pi^-_{\mathcal{P}}(x;\chi) := \#\{p \leqslant x : p \in \mathcal{P} \text{ and } \chi(p) = -1\}.$$

THEOREM 1.3. *Fix $\mathcal{P} \in \mathscr{A}(\delta, \sigma_0, \varepsilon)$. Let χ be a primitive quadratic Dirichlet character, and suppose that*

$$\pi^-_{\mathcal{P}}(x;\chi) = \rho\,\pi^-(x;\chi) + O(x^{\sigma_0+\varepsilon(x)}), \tag{1.4}$$

where $\rho \in (0,1]$. Suppose further that $\rho/\delta = A/B$ for two positive integers A, B. Then, the function $L_{\mathcal{P}}(s,\chi)$ defined by

$$L_{\mathcal{P}}(s,\chi) := \prod_{p\in\mathcal{P}} (1-\chi(p)p^{-s})^{-1/\delta} \qquad (\sigma > 1)$$

extends to a meromorphic function on the region $\{\sigma > \sigma_0\}$, and there is a function $f_{\mathcal{P}}(s,\chi)$ which is analytic on $\{\sigma > \sigma_0\}$ and has the property that

$$\zeta(s)^B L_{\mathcal{P}}(s,\chi)^B = \zeta(s)^A L(s,\chi)^A \exp(f_{\mathcal{P}}(s,\chi)) \qquad (\sigma > \sigma_0).$$

This is proved in §3 below.

1.2. **Remarks.** If one assumes that $\varepsilon(x)$ is such that the integral $\int_1^\infty x^{\varepsilon(x)-1}\,dx$ converges (for example, $\varepsilon(x) := -2(\log\log 2x)/\log x$), then $\zeta_{\mathcal{P}}(s)$ and $f_{\mathcal{P}}(s)$ in Theorem 1.1 extend to continuous functions in the closed half-plane $\{\sigma \geqslant \sigma_0\}$, and the relation (1.3) persists throughout $\{\sigma \geqslant \sigma_0\}$. For such $\varepsilon(x)$ one can easily deduce the following omega result in the case that $\sigma_0 = \frac{1}{2}$.

COROLLARY 1.4. *Let $\kappa : \mathbb{P} \to \{\pm 1\}$ be a function that satisfies the estimate*

$$\#\{\text{prime } p \leqslant x : \kappa(p) = -1\} = \tfrac{1}{2}\pi(x) + O(x^{1/2+\varepsilon(x)}).$$

Then, for any primitive quadratic Dirichlet character χ we have

$$\#\{\text{prime } p \leqslant x : \chi(p) = \kappa(p)\} = \Omega(x^{1/2+\varepsilon(x)}).$$

However, a stronger (and considerably more general) result has been obtained by Kisilevsky and Rubinstein [**12**]. Their work lies much deeper and utilizes explicit information about zeros of L-functions.

1.3. **Examples.** Here, we illustrate the results stated in §1.1 with some special sets of primes.

Let p_n denote the nth smallest prime number for each positive integer n. Note that $n = \pi(p_n)$, thus $\pi(p)$ is the index associated to any given prime p. Let $\mathbb{P}_{k,b}$ denote the set of primes whose index lies in a fixed arithmetic progression

$b \bmod k$; that is,

$$\mathbb{P}_{k,b} := \big\{p \in \mathbb{P} : \pi(p) \equiv b \bmod k\big\}.$$

Let $\mathcal{P} := \mathbb{P}_{k,b}$. Since $p_n \leqslant x$ if and only if $n \leqslant \pi(x)$, we have

$$\pi_{\mathcal{P}}(x) = \#\{n \leqslant \pi(x) : n \equiv b \bmod k\} = \left\lfloor \frac{\pi(x) - b}{k} \right\rfloor = \frac{1}{k}\,\pi(x) + O(1), \qquad (1.5)$$

where $\lfloor\cdot\rfloor$ is the floor function; this shows that (1.1) holds with $\delta = \frac{1}{k}$, $\sigma_0 = 0$, and $\varepsilon(x) \equiv 0$; in other words, $\mathbb{P}_{k,b} \in \mathscr{A}(\frac{1}{k}, 0, 0)$. Applying Theorem 1.1 and Corollary 1.2 we immediately deduce the following.

Corollary 1.5. *The function*

$$\zeta_{k,b}(s) := \prod_{p \in \mathbb{P}_{k,b}} (1 - p^{-s})^{-k} \qquad (\sigma > 1).$$

extends to a meromorphic function on the region $\{\sigma > 0\}$*, and there is a function* $f_{k,b}(s)$ *which is analytic on* $\{\sigma > 0\}$ *and has the property that*

$$\zeta_{k,b}(s) = \zeta(s)\exp(f_{k,b}(s)) \qquad (\sigma > 0).$$

Consequently, the Riemann hypothesis is true if and only if $\zeta_{k,b}(s) \neq 0$ *in* $\{\sigma > \frac{1}{2}\}$.

This shows that much analytic information about the Riemann zeta function (in particular, the location of the nontrivial zeros) is captured by a set of primes of relative density $\frac{1}{k}$.

More generally, for fixed $\kappa, \lambda \in \mathbb{R}$ with $\kappa \geqslant 1$, let $\mathcal{B}_{\kappa,\lambda}$ be the non-homogeneous Beatty sequence defined by

$$\mathcal{B}_{\kappa,\lambda} := \big\{n \in \mathbb{N} : n = \lfloor \kappa m + \lambda \rfloor \text{ for some } m \in \mathbb{Z}\big\}.$$

Beatty sequences appear in a variety of mathematical settings; the arithmetic properties of these sequences have been extensively explored in the literature. Let $\mathbb{P}_{\kappa,\lambda}$ denote the set of primes whose index lies in $\mathcal{B}_{\kappa,\lambda}$; that is,

$$\mathbb{P}_{\kappa,\lambda} := \big\{p \in \mathbb{P} : \pi(p) \in \mathcal{B}_{\kappa,\lambda}\big\}.$$

As with (1.5) above, the estimate

$$\pi_{\mathcal{P}}(x) = \tfrac{1}{\kappa}\pi(x) + O(1)$$

is immediate; therefore, $\mathbb{P}_{\kappa,\lambda} \in \mathscr{A}(\frac{1}{\kappa}, 0, 0)$, and one obtains a natural extension of Corollary 1.5 with the function

$$\zeta_{\kappa,\lambda}(s) := \prod_{p \in \mathbb{P}_{\kappa,\lambda}} (1 - p^{-s})^{-\kappa}.$$

Next, let $\mathbf{X} := \{\mathbf{X}_p : p \in \mathbb{P}\}$ be a set of independent random variables, where each variable is either $+1$ or -1, with a 50% probability for either value. The

law of the iterated logarithm (due to Khintchine **[11]**) asserts that

$$\varlimsup_{x\to\infty} \big(\pi(x)\log\log\pi(x)\big)^{-1/2}\sum_{p\leqslant x}\mathbf{X}_p = \sqrt{2} \qquad \text{a.s.}$$

and (replacing $\{\mathbf{X}_p\}$ with $\{-\mathbf{X}_p\}$) that

$$\varliminf_{x\to\infty} \big(\pi(x)\log\log\pi(x)\big)^{-1/2}\sum_{p\leqslant x}\mathbf{X}_p = -\sqrt{2} \qquad \text{a.s.},$$

where "a.s." stands for "almost surely" in the sense of probability theory. In particular, denoting

$$\mathbb{P}^+_{\mathbf{X}} := \big\{p\in\mathbb{P} : \mathbf{X}_p = +1\big\} \qquad \text{and} \qquad \mathbb{P}^-_{\mathbf{X}} := \big\{p\in\mathbb{P} : \mathbf{X}_p = -1\big\},$$

we have the (less precise) estimate

$$\pi_{\mathbb{P}^+_{\mathbf{X}}}(x) - \pi_{\mathbb{P}^-_{\mathbf{X}}}(x) = \sum_{p\leqslant x}\mathbf{X}_p = O(x^{1/2}) \qquad \text{a.s.}$$

Since $\pi(x) = \pi_{\mathbb{P}^+_{\mathbf{X}}}(x) + \pi_{\mathbb{P}^-_{\mathbf{X}}}(x)$ we deduce that

$$\pi_{\mathbb{P}^\pm_{\mathbf{X}}}(x) = \tfrac{1}{2}\,\pi(x) + O(x^{1/2}) \qquad \text{a.s.}$$

for either choice of the sign $\pm$. Taking $\mathcal{P} := \mathbb{P}^\pm_{\mathbf{X}}$ we see that (1.1) holds a.s. with $\delta = \sigma_0 = \frac{1}{2}$ and $\varepsilon(x) \equiv 0$; in other words, $\mathbb{P}^\pm_{\mathbf{X}} \in \mathscr{A}(\frac{1}{2}, \frac{1}{2}, 0)$ almost surely. In view of Theorem 1.1 and Corollary 1.2 we deduce the following.

Corollary 1.6. *In the region $\{\sigma > 1\}$, let*

$$\zeta^+_{\mathbf{X}}(s) := \prod_{\substack{p\in\mathbb{P}\\ \mathbf{X}_p=+1}} (1-p^{-s})^{-2} \qquad \text{and} \qquad \zeta^-_{\mathbf{X}}(s) := \prod_{\substack{p\in\mathbb{P}\\ \mathbf{X}_p=-1}} (1-p^{-s})^{-2} \tag{1.6}$$

Then, almost surely, both functions $\zeta^\pm_{\mathbf{X}}(s)$ extend to meromorphic functions on the region $\{\sigma > \frac{1}{2}\}$, and there are functions $f^\pm_{\mathbf{X}}(s)$ which are analytic on $\{\sigma > \frac{1}{2}\}$ and are such that

$$\zeta^\pm_{\mathbf{X}}(s) = \zeta(s)\exp(f^\pm_{\mathbf{X}}(s)) \qquad (\sigma > \tfrac{1}{2}). \tag{1.7}$$

The Riemann hypothesis is equivalent to the assertion that, almost surely, $\zeta^\pm_{\mathbf{X}}(s) \neq 0$ in $\{\sigma > \frac{1}{2}\}$ for either choice of the sign $\pm$.

In the region $\{\sigma > 1\}$, let us now define

$$L(s,\mathbf{X}) := \prod_{p\in\mathbb{P}}(1-\mathbf{X}_p\,p^{-s})^{-1}. \tag{1.8}$$

The next corollary reproduces a result that was first proved by Wintner **[25]** and laid the foundation for random multiplicative function theory; it asserts that the GRH almost surely holds for the "L-function" $L(s,\mathbf{X})$ (for more modern work in this direction, see **[3, 4, 7, 8, 13]**).

Corollary 1.7. *The function $L(s,\mathbf{X})$ almost surely extends to an analytic function without zeros in the region $\{\sigma > \frac{1}{2}\}$.*

Indeed, using (1.6) and (1.8) we have

$$\begin{aligned} L(s,\mathbf{X})^2 &= \prod_{p\in\mathbb{P}} (1 - \mathbf{X}_p\, p^{-s})^{-2} \\ &= \prod_{p\in\mathbb{P}_{\mathbf{X}}^+} (1+p^{-s})^{-2} \prod_{p\in\mathbb{P}_{\mathbf{X}}^-} (1-p^{-s})^{-2} \\ &= \prod_{p\in\mathbb{P}_{\mathbf{X}}^+} (1-p^{-2s})^{-2} \prod_{p\in\mathbb{P}_{\mathbf{X}}^+} (1-p^{-s})^{2} \prod_{p\in\mathbb{P}_{\mathbf{X}}^-} (1-p^{-s})^{-2} \\ &= \zeta_{\mathbf{X}}^+(2s)\zeta_{\mathbf{X}}^+(s)^{-1}\zeta_{\mathbf{X}}^-(s), \end{aligned}$$

By Corollary 1.6 there are (almost surely) functions $f_{\mathbf{X}}^{\pm}(s)$ which are analytic on $\{\sigma > \frac{1}{2}\}$ and satisfy (1.7); in particular, the relation

$$L(s,\mathbf{X})^2 = \zeta_{\mathbf{X}}^+(2s)\zeta_{\mathbf{X}}^+(s)^{-1}\zeta_{\mathbf{X}}^-(s) = \zeta(2s)\exp\big(f_{\mathbf{X}}^+(2s) - f_{\mathbf{X}}^+(s) + f_{\mathbf{X}}^-(s)\big)$$

holds in $\{\sigma > 1\}$, and it provides the required analytic continuation of $L(s,\mathbf{X})$ to the region $\{\sigma > \frac{1}{2}\}$. Moreover, $L(s,\mathbf{X}) \ne 0$ in $\{\sigma > \frac{1}{2}\}$.

1.4. **Asymptotic additive bases.**

Theorem 1.8. *Every set $\mathcal{P} \in \mathscr{B}(\delta,\varepsilon)$ containing the prime 2 is an exact asymptotic additive basis for $\mathbb{N}$. In other words, there is an integer $h = h(\mathcal{P}) > 0$ such that the h-fold sumset*

$$h\mathcal{P} := \underbrace{\mathcal{P} + \cdots + \mathcal{P}}_{h \text{ copies}}$$

contains all but finitely many natural numbers.

This is proved in §4 below. We remark that Sárközy **[20]** has shown that any set of primes $\mathcal{P}$ is an asymptotic additive basis for $\mathbb{N}$, and stronger quantitative versions have been obtained; see **[14,15,18,21]**. To show that every $\mathcal{P} \in \mathscr{B}(\delta,\varepsilon)$ containing 2 is an *exact* asymptotic additive basis, we use a result of Shiu **[22]** on strings of consecutive primes in an arithmetic progression; in principle, the methods of Green and Tao **[5]** could be used to prove Theorem 1.8 with $\mathscr{B}(\delta,\varepsilon)$ replaced with a rather more restricted class of prime sets.

1.5. **Examples.**

As in §1.3, we put

$$\mathbb{P}_{k,b} := \big\{p \in \mathbb{P} : \pi(p) \equiv b \bmod k\big\}.$$

We have already seen that

$$\pi_{\mathcal{P}}(x) = \tfrac{1}{k}\pi(x) + O(1)$$

holds with $\mathcal{P} := \mathbb{P}_{k,b}$, and therefore $\mathbb{P}_{k,b} \in \mathscr{B}(\frac{1}{k}, 0)$. Since $2 \in \mathbb{P}_{k,b}$ if and only if $b = 1$, the next corollary follows immediately from Theorem 1.8.

Corollary 1.9. *For every $k \in \mathbb{N}$, the set $\mathbb{P}_{k,1}$ is an exact asymptotic additive basis for $\mathbb{N}$. For all $b, k \in \mathbb{N}$, the set $\mathbb{P}_{k,b} \cup \{2\}$ is an exact asymptotic additive basis for $\mathbb{N}$.*

For example, an exact asymptotic additive basis for $\mathbb{N}$ is provided by the set

$$\mathbb{P}_{100,1} = \{2, 547, 1229, 1993, 2749, 3581, 4421, 5281 \ldots\},$$

which consists of 2 and every hundredth prime thereafter.

More generally, for the set $\mathbb{P}_{\kappa,\lambda}$ defined in §1.3, we have the following result.

Corollary 1.10. *For any $\kappa, \lambda \in \mathbb{R}$ with $\kappa \geqslant 1$, the set $\mathbb{P}_{\kappa,\lambda} \cup \{2\}$ is an exact asymptotic additive basis for $\mathbb{N}$.*

2. Proof of Theorem 1.1

Suppose first that $s \in \mathbb{C}$ with $\sigma > 1$. From the Euler product representations of $\zeta_{\mathcal{P}}(s)$ and $\zeta(s)$ we see that the function

$$f_{\mathcal{P}}(s) := \log \zeta_{\mathcal{P}}(s) - \log \zeta(s)$$

can be written in the form

$$f_{\mathcal{P}}(s) = \sum_{j \geqslant 1} j^{-1} f_{\mathcal{P},j}(s)$$

with

$$f_{\mathcal{P},j}(s) := \frac{1}{\delta} \sum_{p \in \mathcal{P}} p^{-js} - \sum_{p \in \mathbb{P}} p^{-js} \qquad (j \geqslant 1). \tag{2.1}$$

To prove the theorem, it is enough to show that $f_{\mathcal{P}}(s)$ extends to an analytic function in $\{\sigma > \sigma_1\}$ for every real number $\sigma_1 > \sigma_0$.

Let σ_1 be given. Noting that $\sigma_1 > 0$, let N be a positive integer such that $\sigma_1 > \frac{1}{N}$. It is easy to verify that

$$\sum_{j > N} j^{-1} f_{\mathcal{P},j}(s)$$

extends to an analytic function in $\{\sigma > \frac{1}{N}\}$, hence also in $\{\sigma > \sigma_1\}$. Therefore, it remains to show that for any fixed $j \in [1, N]$, $f_{\mathcal{P},j}(s)$ extends to an analytic function in $\{\sigma > \sigma_1\}$.

Using (1.1) we have

$$\frac{1}{\delta} \pi_{\mathcal{P}}(u) = \pi(u) + E(u) \qquad (u \geqslant 1),$$

where $E(u) \ll u^{\sigma_0+\varepsilon(u)}$, and therefore

$$\begin{aligned}\frac{1}{\delta}\sum_{p\in\mathcal{P}} p^{-js} &= \frac{1}{\delta}\int_1^\infty u^{-js}\,d\pi_{\mathcal{P}}(u) = \frac{js}{\delta}\int_1^\infty u^{-js-1}\pi_{\mathcal{P}}(u)\,du\\ &= js\int_1^\infty u^{-js-1}\pi(u)\,du + js\int_1^\infty u^{-js-1}E(u)\,du\\ &= \sum_{p\in\mathbb{P}} p^{-js} + js\int_1^\infty u^{-js-1}E(u)\,du;\end{aligned}$$

that is,

$$f_{\mathcal{P},j}(s) = js\int_1^\infty u^{-js-1}E(u)\,du.$$

Since $E(u) \ll u^{\sigma_0+\varepsilon(u)}$, the latter integral converges absolutely in $\{\sigma > j^{-1}\sigma_0\}$, hence also in $\{\sigma > \sigma_1\}$, and the integral representation provides the required analytic extension of $f_{\mathcal{P},j}(s)$ when $j \in [1, N]$.

3. Proof of Theorem 1.3

As in §2 we first assume that $s \in \mathbb{C}$ with $\sigma > 1$ and define

$$f_{\mathcal{P}}(s,\chi) := B\log(\zeta(s)L_{\mathcal{P}}(s,\chi)) - A\log(\zeta(s)L(s,\chi)) = \sum_{j\geqslant 1} j^{-1} f_{\mathcal{P},j}(s,\chi),$$

where

$$f_{\mathcal{P},j}(s,\chi) := \frac{B}{\delta}\sum_{p\in\mathcal{P}}\chi(p)^j p^{-js} - A\sum_{p\in\mathbb{P}}\chi(p)^j p^{-js} + (B-A)\sum_{p\in\mathbb{P}} p^{-js} \qquad (j \geqslant 1).$$

As before, let $\sigma_1 > \sigma_0$ be given, and let N be a fixed positive integer such that $\sigma_1 > \frac{1}{N}$. To prove the theorem, it is enough to show that for any fixed $j \in [1, N]$ the function $f_{\mathcal{P},j}(s,\chi)$ has an analytic extension to the region $\{\sigma > \sigma_1\}$.

Put

$$\begin{aligned} f_1(s) &:= \frac{B}{\delta}\sum_{\substack{p\in\mathcal{P}\\ \chi(p)=1}} p^{-js} - A\sum_{\substack{p\in\mathbb{P}\\ \chi(p)=1}} p^{-js},\\ f_2(s) &:= \frac{B}{\delta}\sum_{\substack{p\in\mathcal{P}\\ \chi(p)=-1}} p^{-js} - A\sum_{\substack{p\in\mathbb{P}\\ \chi(p)=-1}} p^{-js},\\ f_3(s) &:= \frac{B}{\delta}\sum_{\substack{p\in\mathcal{P}\\ p\mid q}} p^{-js} - A\sum_{\substack{p\in\mathbb{P}\\ p\mid q}} p^{-js},\\ f_4(s) &:= (B-A)\sum_{p\in\mathbb{P}} p^{-js},\end{aligned}$$

where q is the modulus of the character χ. We have

$$f_1(s) + f_2(s) + f_3(s) + f_4(s) = \frac{B}{\delta}\sum_{p\in\mathcal{P}} p^{-js} - B\sum_{p\in\mathbb{P}} p^{-js} = B\,f_{\mathcal{P},j}(s),$$

where $f_{\mathcal{P},j}(s)$ is given by (2.1). Recall that in §2 we have shown that $f_{\mathcal{P},j}(s)$ has an analytic extension to the region $\{\sigma > \sigma_1\}$; the same is also true of $f_3(s)$ (which is clearly entire). Now observe that

$$f_{\mathcal{P},j}(s,\chi) = f_1(s) + (-1)^j f_2(s) + f_4(s),$$

and therefore

$$f_{\mathcal{P},j}(s,\chi) = \begin{cases} -f_3(s) + B\, f_{\mathcal{P},j}(s) & \text{if } j \text{ is even,} \\ 2f_2(s) - f_3(s) + B\, f_{\mathcal{P},j}(s) & \text{if } j \text{ is odd.} \end{cases}$$

To conclude the proof it remains to show that $f_2(s)$ extends analytically to the region $\{\sigma > \sigma_1\}$.

Since $\rho/\delta = A/B$ we have

$$f_2(s) = \frac{A}{\rho} \sum_{\substack{p\in\mathcal{P}\\ \chi(p)=-1}} p^{-js} - A \sum_{\substack{p\in\mathbb{P}\\ \chi(p)=-1}} p^{-js}.$$

Using (1.4) we can write

$$\frac{A}{\rho}\, \pi_{\mathcal{P}}(u,\chi) = A\, \pi(u,\chi) + E(u),$$

where $E(u) \ll u^{\sigma_0+\varepsilon(u)}$. Then

$$\begin{aligned} \frac{A}{\rho} \sum_{\substack{p\in\mathcal{P}\\ \chi(p)=-1}} p^{-js} &= \frac{A}{\rho} \int_1^\infty u^{-js}\, d\pi_{\mathcal{P}}^-(u,\chi) = \frac{jsA}{\rho} \int_1^\infty u^{-js-1} \pi_{\mathcal{P}}^-(u,\chi)\, du \\ &= jsA \int_1^\infty u^{-js-1}\pi^-(u,\chi)\, du + js \int_1^\infty u^{-js-1} E(u)\, du \\ &= A \sum_{\substack{p\in\mathbb{P}\\ \chi(p)=1}} p^{-js} + js \int_1^\infty u^{-js-1} E(u)\, du; \end{aligned}$$

in other words,

$$f_2(s) = js \int_1^\infty u^{-js-1} E(u)\, du.$$

Since $E(u) \ll u^{\sigma_0+\varepsilon(u)}$, the integral representation yields the desired analytic continuation of $f_2(s)$ to $\{\sigma > \sigma_1\}$.

4. Proof of Theorem 1.8

For the proof of Theorem 1.8, we use the following result of Banks, Güloğlu and Vaughan [**2**, Theorem 1.2] (the proof of which relies a deep theorem of Kneser; see Halberstam and Roth [**6**, Chapter I, Theorem 18]).

Lemma 4.1. *Let $\mathcal{P}$ be a set of prime numbers such that*

$$\varliminf_{x\to\infty} \frac{\pi_{\mathcal{P}}(x)}{x/\log x} > 0.$$

Suppose that there is a number s_1 such that for all $s \geqslant s_1$ and $a, b \in \mathbb{N}$, the congruence

$$p_1 + \cdots + p_s \equiv a \bmod b$$

has a solution with $p_1, \ldots, p_s \in \mathcal{P}$. Then, there is an integer $h = h(\mathcal{P}) > 0$ such that the h-fold sumset $h\mathcal{P}$ contains all but finitely many natural numbers.

We also use the following statement concerning consecutive primes in a given arithmetic progression, which is due to Shiu [**22**, Theorem 1]; see also Banks *et al* [**1**, Corollary 4], where a bounded gaps variant is obtained as a consequence of the Maynard-Tao theorem (see [**16**]).

LEMMA 4.2. *Let p_n denote the nth smallest prime number for each positive integer n. Fix $c, d \in \mathbb{N}$ with $\gcd(c, d) = 1$. Then, there are infinitely many $r \in \mathbb{N}$ such that $p_{r+1} \equiv p_{r+2} \equiv \cdots \equiv p_{r+m(r)} \equiv c \bmod d$, where $m(r)$ is an integer-valued function satisfying the lower bound*

$$m(r) \gg \left(\frac{\log\log r \ \log\log\log\log r}{(\log\log\log r)^2}\right)^{1/\phi(d)}. \tag{4.1}$$

Here, $\phi(\cdot)$ is the Euler function.

We now make an important observation based on Lemma 4.2, which may be of independent interest.

PROPOSITION 4.3. *Fix $\mathcal{P} \in \mathscr{B}(\delta, \varepsilon)$. For all $c, d \in \mathbb{N}$ with $\gcd(c, d) = 1$, the set $\mathcal{P}$ contains infinitely many primes in the arithmetic progression $c \bmod d$.*

Proof. According to Lemma 4.2, there is an infinite set $\mathcal{S} \subseteq \mathbb{N}$ with the property that

$$p_{r+1} \equiv p_{r+2} \equiv \cdots \equiv p_{r+m(r)} \equiv c \bmod d \qquad (r \in \mathcal{S}), \tag{4.2}$$

where $m(r)$ satisfies (4.1). Taking into account (1.2), we derive the following estimate for all $r \in \mathcal{S}$:

$$\begin{aligned}\pi_{\mathcal{P}}(p_{r+m(r)}) - \pi_{\mathcal{P}}(p_r) &= \delta\big(\pi(p_{r+m(r)}) - \pi(p_r)\big) + O\big((\log\log r)^{\varepsilon_r}\big) \\ &= \delta\, m(r) + O\big((\log\log r)^{\varepsilon_r}\big)\end{aligned}$$

where $\varepsilon_r := \varepsilon(p_{r+m(r)})$ for each r, and the constant implied by the O-symbol depends only on $\mathcal{P}$. In view of (4.1) and the fact that $\overline{\lim\limits_{r\to\infty}}\, \varepsilon_r \leqslant 0$, we have

$$\pi_{\mathcal{P}}(p_{r+m(r)}) > \pi_{\mathcal{P}}(p_r) \qquad (r \in \mathcal{S},\ r \geqslant r_0). \tag{4.3}$$

For every sufficiently large $r \in \mathcal{S}$, by (4.3) it follows that $p_{r+j} \in \mathcal{P}$ for some j in the range $1 \leqslant j \leqslant m(r)$, and by (4.2) we have $p_{r+j} \equiv c \bmod d$. Since $\mathcal{S}$ is infinite, the lemma follows. $\square$

Using the Hardy-Littlewood circle method, Vinogradov [**24**] established his famous theorem that every sufficiently large odd integer is the sum of three prime numbers. Effective versions of Vinogradov's theorem have been given by several authors (see [**10**,**17**,**23**] and references therein), but for the purposes of the present paper we require only the following extension of Vinogradov's theorem, which is due to Haselgrove [**9**, Theorem A].

LEMMA 4.4. *For any fixed $\theta \in (\frac{63}{64}, 1)$ there is a positive number $n_0(\theta)$ such that every odd integer $n \geqslant n_0(\theta)$ can be expressed as the sum of three primes*

$$n = p_1 + p_2 + p_3$$

with $|p_j - \frac{1}{3}n| < n^\theta$ for each $j = 1, 2, 3$.

The following statement is a simple consequence of Haselgrove's result.

LEMMA 4.5. *For every integer $s \geqslant 6$, there is an integer $N_0(s)$ with the property that every integer $N \geqslant N_0(s)$ can be expressed as a sum of primes*

$$N = \widetilde{p}_1 + \cdots + \widetilde{p}_s$$

with $\widetilde{p}_j = 2$ or $\widetilde{p}_j \geqslant \frac{1}{12}N$ for $j = 1, \ldots, s$.

Proof. Set $\theta := \frac{99}{100}$. Since $\theta \in (\frac{63}{64}, 1)$, Lemma 4.4 shows that there is a positive number $n_0 = n_0(\theta)$ such that every odd integer $n \geqslant n_0$ can be expressed as the sum of three primes, $n = p_1 + p_2 + p_3$, with $p_j \geqslant \frac{1}{4}n$ for each $j = 1, 2, 3$.

Put $N_1(s) := n_0 + 2s - 6$, and let N be an odd integer exceeding $N_1(s)$. Since $n := N - 2s + 6$ is an odd integer exceeding n_0, we can write $n = p_1 + p_2 + p_3$ as above. Consequently,

$$N = p_1 + p_2 + p_3 + \underbrace{2 + \cdots + 2}_{s-3 \text{ copies}}$$

where

$$p_j \geqslant \tfrac{1}{4}n = \tfrac{1}{4}(N - 2s + 6) \qquad (j = 1, 2, 3). \tag{4.4}$$

Replacing $N_1(s)$ by a larger number, if necessary, the bound $N > N_1(s)$ and (4.4) together imply that $p_j \geqslant \frac{1}{12}N$ for $j = 1, 2, 3$.

Next, put $N_2(s) := 3n_0 + 6s - 36$, and let N be an even integer exceeding $N_2(s)$. If n_0 is sufficiently large (which we can assume) then $N - 6s + 36 = n + n'$ for some odd integers n and n' that are both larger than $\max\{n_0, \frac{1}{3}N\}$. Therefore, writing $n = p_1 + p_2 + p_3$ and $n' = p_1' + p_2' + p_3'$ as above, we have

$$N = p_1 + p_2 + p_3 + p_1' + p_2' + p_3' + \underbrace{2 + \cdots + 2}_{s-6 \text{ copies}}$$

where

$$p_j \geqslant \tfrac{1}{4}n \geqslant \tfrac{1}{12}N \qquad \text{and} \qquad p_j' \geqslant \tfrac{1}{4}n' \geqslant \tfrac{1}{12}N \qquad (j = 1, 2, 3).$$

Taking $N_0(s) := \max\{N_1(s), N_2(s)\}$ we finish the proof. $\square$

Proof of Theorem 1.8. Fix a set $\mathcal{P} \in \mathscr{B}(\delta, \varepsilon)$ with $2 \in \mathcal{P}$. Since $\pi_{\mathcal{P}}(x)$ satisfies (1.2) the first condition of Lemma 4.1 is met, and it remains only to verify the second condition of Lemma 4.1.

Fix an arbitrary integer $s \geqslant 6$, and let $a, b \in \mathbb{N}$ be given. Replacing a with a sufficiently large number in the progression $a \bmod b$, we can assume that $a \geqslant 24b$. We can further assume that a exceeds the number $N_0(s)$ described in the statement of Lemma 4.5. Therefore, a can be expressed as a sum of primes $a = \widetilde{p}_1 + \cdots + \widetilde{p}_s$, where for every $j = 1, \ldots, s$ we have either $\widetilde{p}_j = 2$ or else

$$\widetilde{p}_j \geqslant \tfrac{1}{12} a \geqslant 2b > b.$$

In the latter case, it is clear that $\gcd(\widetilde{p}_j, b) = 1$, hence by Proposition 4.3 there is a prime $p_j \in \mathcal{P}$ such that

$$p_j \equiv \widetilde{p}_j \bmod b. \tag{4.5}$$

Since $2 \in \mathcal{P}$, we can put $p_j := 2$ whenever $\widetilde{p}_j = 2$, obtaining (4.5) in this case as well. Summing the congruences (4.5) over $j = 1, \ldots, s$ gives

$$a = \widetilde{p}_1 + \cdots + \widetilde{p}_s \equiv p_1 + \cdots + p_s \bmod b.$$

This shows that the second condition of Lemma 4.1 is met, and the proof of Theorem 1.8 is complete. $\square$

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References

[1] W. D. Banks, T. Freiberg and C. L. Turnage-Butterbaugh, "Consecutive primes in tuples," *Acta Arith.* **167** (2015), no. 3, 261–266.

[2] W. D. Banks, A. M. Güloğlu and R. C. Vaughan, "Waring's problem for Beatty sequences and a local to global principle," *J. Théor. Nombres Bordeaux* **26** (2014), no. 1, 1–16.

[3] S. Chatterjee and K. Soundararajan, "Random multiplicative functions in short intervals," *Int. Math. Res. Not.* (2012), no. 3, 479–492.

[4] A. Granville and K. Soundararajan, "The distribution of values of $L(1, \chi_d)$," *Geom. Funct. Anal.* **13** (2003), no. 5, 992–1028.

[5] B. Green and T. Tao, "Linear equations in primes," *Ann. of Math. (2)* 171 (2010), no. 3, 1753–1850.

[6] H. Halberstam and K. F. Roth, *Sequences*. Second edition. Springer-Verlag, New York-Berlin, 1983.

[7] A. J. Harper, "On the limit distributions of some sums of a random multiplicative function," *J. Reine Angew. Math.* **678** (2013), 95–124.

[8] A. J. Harper, A. Nikeghbali and M. Radziwiłł, "A note on Helson's conjecture on moments of random multiplicative functions," preprint, 2015. `arXiv:1505.01443`

[9] C. B. Haselgrove, "Some theorems in the analytic theory of numbers," *J. London Math. Soc.* **26** (1951), 273–277.

[10] H. A. Helfgott, "Chebotarev sets," 2011. Preprint available on arXiv: `arXiv:1312.7748`

[11] A. Khintchine, "Über einen Satz der Wahrscheinlichkeitsrechnung," *Fund. Math.* **6** (1924), 9–20.

[12] H. Kisilevsky and M. O. Rubinstein, "Chebotarev sets," 2011. Preprint available on arXiv: `arXiv:1112.4945`

[13] Y.-K. Lau, G. Tenenbaum and J. Wu, "On mean values of random multiplicative functions," *Proc. Amer. Math. Soc.* **141** (2013), no. 2, 409–420.

[14] H. Li and H. Pan, "A density version of Vinogradov's three primes theorem," *Forum Math.* 22 (2010), no. 4, 699–714.

[15] K. Matomäki, "Sums of positive density subsets of the primes," *Acta Arith.* 159 (2013), no. 3, 201–225.

[16] J. Maynard, "Small gaps between primes," *Ann. of Math.* (2) **181** (2015), no. 1, 383–413.

[17] O. Ramaré, "On Šnirel'man's constant," *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **22** (1995), no. 4, 645–706.

[18] O. Ramaré and I. Z. Ruzsa, "Additive properties of dense subsets of sifted sequences," *J. Théor. Nombres Bordeaux* 13 (2001), no. 2, 559–581.

[19] B. Riemann, "Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse," *Monatsberichte der Berliner Akademie*, 1859.

[20] A. Sárközy, "On finite addition theorems. Structure theory of set addition." *Astérisque* 258 (1999), nos. xi-xii, 109–127.

[21] X. Shao, "A density version of the Vinogradov three primes theorem," *Duke Math. J.* 163 (2014), no. 3, 489–512.

[22] D. K. L. Shiu, "Strings of congruent primes," *J. London Math. Soc. (2)* **61** (2000), no. 2, 359–373.

[23] T. Tao, "Every odd number greater than 1 is the sum of at most five primes," *Math. Comp.* **83** (2014), no. 286, 997–1038.

[24] I. M. Vinogradov, "Representation of an odd number as a sum of three primes," *Comptes Rendus (Doklady) de l'Academy des Sciences de l'USSR* **15** (1937), 191–294.

[25] A. Wintner, "Random factorizations and Riemann's hypothesis," *Duke Math. J.* **11** (1944), 267–275.

Department of Mathematics, University of Missouri, Columbia MO, USA.
E-mail address: `bankswd@missouri.edu`